Buckling of Unbalanced Anisotropic Sandwich Plates with Finite Bonding Stiffness

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The buckling problem of a simply supported unbalanced sandwich plate is solved using the Rayleigh-Ritz method. The sandwich plate consists of an antiplane core, two unequal anisotropic laminated faces, and adhesive layers of finite bonding stiffness. The analysis takes into account all the possible coupling actions in faces to examine the coupling effects of sandwich buckling behavior. The buckling loads are calculated for various face lay-up angles, core to face thickness ratios, and aspect ratios. The bonding stiffness is quite critical for a sandwich plate with thick faces and stiff core when it is smaller than the threshold value.

Nomenclature

a,b	= plate dimensions in x and y directions, respectively
A_{ij},B_{ij},D_{ij}	= face stretching, bending-stretching coupling, and stiffnesses, respectively
c	= thickness of core
C_r	= core properties ratio
C_{mn}^i	= Fourier coefficient
d	= face thickness when $d_1 = d_2$
d_1,d_2	= thicknesses of upper and lower faces, respectively
E	= core transverse normal stiffness
E_1,E_2	= face lamina modulus
G_x,G_y	= core shear stiffness
G_{12}	= face shear modulus
h	$= c + d_1/2 + d_2/2$
h_1	$= d_1/2 + c/2$
h_2	$= d_2/2 + c/2$
K	= adhesive bonding stiffness when $K_x - K_y$
K_x,K_y	= adhesive bonding stiffness in x and y directions, respectively
m,n	= summation variables
N	= number of layers in face
N_x,N_y	= total inplane force resultants in x and y directions, respectively
N_{xy}	= total inplane shear force resultant
u,v,w	= displacement components
U_1,U_2	= face contributions to the strain energy
U_3,U_4	= core contributions to the strain energy
V	= external work
x,y,z	= coordinates
α_{mn}	$= m\pi/a$
β_{mn}	$= n\pi/b$
γ_x,γ_y	= shear strain components of core
θ	= face lay-up angle with respect to the x axis
λ_x,λ_y	= Lagrange multipliers
Π	= total potential energy
ϕ_x,ϕ_y	= compatibility equations in x and y directions, respectively

Superscripts

*	= normalizing quantity
0	= orthotropic solution
1,2	= upper and lower faces, respectively

Subscripts

0	= value at $z = 0$
1,2	= upper and lower faces, respectively
x,y	= x and y directions, respectively

Introduction

SANDWICH structures are constructed of various materials for face and core by bonding them together through adhesives to meet structural and environmental requirements. Sandwich structure is attractive in the aerospace industries because of its high specific bending stiffness. The honeycomb core is light but stiff enough to maintain the distance between faces. The bonding of stiff faces and soft core for the sandwich is very important in fabrication. Adhesive films are often used for bonding. The adhesives are generally synthetic polymers such as expoxies, polyimides, urethanes, phenolics, etc. The polymeric materials show considerable material degradation due to their viscoelastic nature at elevated service temperature.

Many theories of bending and stability of flat sandwich plates have various assumptions differing from each other, especially regarding the methods of treating flexural rigidities of two facings about their own midplanes, material anisotropy, transverse shear and normal deformations of core, and the face-core interface conditions.

Eringen[1] incorporated the previously omitted effects of the flexural rigidity as well as transverse normal deformation of the core into the sandwich theories with usual assumptions that the faces are thin structures with or without flexural rigidity and the core undergoes only transverse shear deformation. Benson and Mayers[2] investigated the general instability and face wrinkling through variational procedure based on potential energy and the use of Lagrange multipliers to introduce face-core continuity. They handled the stability modes of sandwich plates with two identical isotropic faces and ideally orthotropic honeycomb core, including the transverse normal strain of core. Hussein[3] introduced the nonrigid bonding effects into the sandwich theory. The theory handles sandwich plates with two isotropic identical faces, isotropic core with no inplane shear and transverse normal strain, and the adhesive layers of finite bonding stiffness under the transverse loads.

From the point of view of material anisotropy, most previous papers on sandwich plates consisting of laminated faces assume that the principal material axes are parallel to the geometric axes, i.e., are orthotropic. However, the material shows anisotropic behavior when the material axes are arbitrarily oriented. Analysis of sandwich plates with anisotropic faces is more complicated than that with orthotropic faces. The complexity in the analysis of sandwich plates with anisotropic laminated faces is due to the presence of three types of asymmetry resulting from the lay-up sequences in the faces: 1) the asymmetry with respect to the midplane of the face (face

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asymmetry) induces face bending-stretching coupling action; 2) the asymmetry with respect to the midplane of the core (plate asymmetry) induces the plate bending-stretching coupling action; and 3) the material asymmetry of faces and core with respect to the geometric axes of the plate introduces coupling between the stretching and shearing action.

There have been works dealing with the buckling behavior of anisotropic laminates that show only limited coupling actions.[4-6] Chang et al.[7] investigated the buckling of orthotropic facing sandwich plates. Librescu[8] presented one of the first formulations of theory of anisotropic facing sandwich plates. Monforton and Ibrahim[9,10] suggested a modified stiffness formulation for predicting the flexural deflection and stresses of unsymmetric anisotropic sandwich plates with bending-stretching coupling. They concluded that the neglect of bending-stretching coupling in the case of unbalanced FRP-faced sandwich plates underestimates the deflection and the internal forces. These references did not consider the coupling of stretching-shearing action.

Rao[11,12] extended the small-deflection theory of orthotropic sandwich plates developed by Libove and Batdorf[13] to highly anisotropic sandwich plates with thin faces. He demonstrated the effect of stretching-shearing coupling action on the stability of anisotropic laminated flat sandwich plates[11] and cylindrically curved sandwich plates[12] subjected to normal forces without considering the face and plate asymmetry and the face bending stiffnesses.

A cursory look at the extensive literature reveals that none of the publications consider all three types of asymmetry in sandwich plates. However, the design versatility in layered sandwich structures requires the generalized analysis of the sandwich plates. The objective of this paper is to investigate the stability of sandwich plates under inplane normal and shear forces considering all three types of asymmetry, honeycomb core with transverse normal and shear deformations, and the nonrigid bonding stiffness of adhesive layers.

Elastic Analysis

The geometry and loadings of the simply supported sandwich plate considered herein are shown in Fig. 1. Four types of sandwich plates are possible according to the lay-up sequences of the faces as shown in Fig. 2. For the general formulation of a sandwich plate, the unbalanced anisotropic sandwich plate with unsymmetric faces (*d* in Fig. 2) is considered.

For the elastic analysis of sandwich plates, the following assumptions are made:

1) Deformations are small, and all constituent materials are homogeneous and linear elastic.

2) The adhesive shear strain is proportional to the face-core interface shear stress.

3) The core is under the antiplane stress state and fully supported along the edges.

The transverse normal modulus of core is much lower than that of the face, so that the transverse normal deformation of the face is negligible in comparison with that of the core. Since as the core is under an antiplane stress state ($\sigma_x = \sigma_y = \sigma_{xy} = 0$), the displacement components of the core u, v, and w are obtained from the equilibrium equations and strain-displacement relations, and are as follows:[14]

$$w(x,y,z) = -z^2(G_x\gamma_{x,x} + G_y\gamma_{y,y})/2E + z\sigma_0/E + w_0 \quad (1)$$

$$u(x,y,z) = \gamma_x + z^3(G_x\gamma_{x,x} + G_y\gamma_{y,y}),_x/6E - z^2\sigma_{0,x}/2E - zw_{0,x} + u_0 \quad (2)$$

$$v(x,y,z) = \gamma_y + z^3(G_x\gamma_{x,x} + G_y\gamma_{y,y}),_y/6E - z^2\sigma_{0,y}/2E - zw_{0,y} + v_0 \quad (3)$$

where σ_0, w_0, u_0, v_0 represent $\sigma_z(x,y,0)$, $w(x,y,0)$, $u(x,y,0)$, and $v(x,y,0)$, respectively. The whole displacement fields of

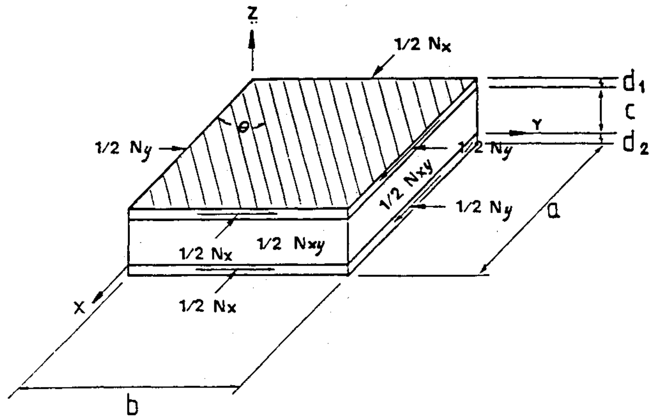

Fig. 1 Geometry and loadings of a sandwich plate.

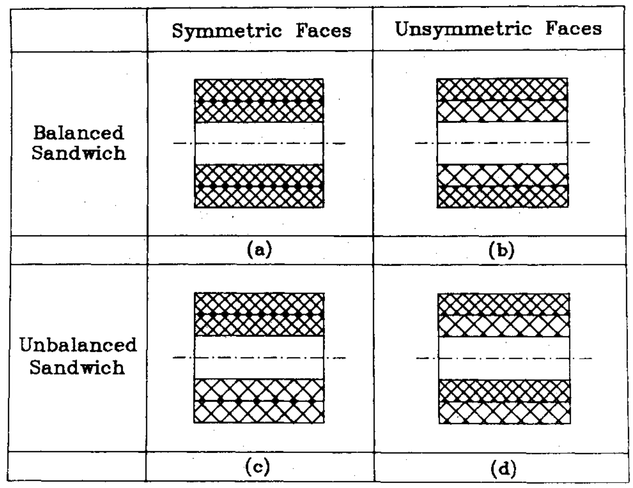

Fig. 2 Four types of sandwich plates with different lay-up sequences.

core are expressed in terms of $\gamma_x(x,y)$, $\gamma_y(x,y)$, $\sigma_0(x,y)$, $w_0(x,y)$, $u_0(x,y)$, $v_0(x,y)$, and the z coordinate. The face-core interface compatibility requirements yield four equations [Eqs. (4-7)] from the assumption that the adhesive shear strain is proportional to the face-core interface shear stress and two equations [Eqs. (8 and 9)] from the continuity of transverse normal displacement w at the face-core interface:

$$u_1 + d_1w_{1,x}/2 - c\gamma_x - c^3(G_x\gamma_{x,x} + G_y\gamma_{y,y}),_x/6E + c^2\sigma_{0,x}/2E + cw_{0,x} - u_0 - G_x\gamma_x/K_x = 0 \quad (4)$$

$$u_0 - u_2 + d_2w_{0,x}/2 - G_x\gamma_x/K_x = 0 \quad (5)$$

$$v_1 + d_1w_{1,y}/2 - c\gamma_y - c^3(G_y\gamma_{x,x} + G_y\gamma_{y,y}),_y/6E + c^2\sigma_{0,y}/2E + cw_{0,y} - v_0 - G_y\gamma_y/K_y = 0 \quad (6)$$

$$v_0 - v_2 + d_2w_{0,y}/2 - G_y\gamma_y/K_y = 0 \quad (7)$$

$$w_1 + c^2(G_x\gamma_{x,x} + G_y\gamma_{y,y})/2E - c\sigma_0/E - w_0 = 0 \quad (8)$$

$$w_2 - w_0 = 0 \quad (9)$$

The bonding stiffnesses K_x and K_y are defined as

$$K_x = SxG_a/t_a \quad (10)$$

$$K_y = S_yG_a/t_a \quad (11)$$

where G_a denotes adhesive shear stiffness, t_a denotes adhesive thickness, and S_x and S_y are bonding correction factors for the honeycomb core in the x and y directions, respectively. For uniform bonding such as in a sandwich plate with a homogeneous core, the bonding stiffness can easily be defined as G_a/t_a, i.e., $S_x = S_y = 1$. But the bonding line in a honeycomb sandwich plate is not so uniform throughout the plate due to the fillet forming of adhesive around the honeycomb cell wall that it is required to define the appropriate correction factors S_x and S_y in the honeycomb bonding.

The displacement components are denoted by u_1, v_1, and w_1 for the upper face and u_2, v_2, w_2 for the lower face. Elimination of u_0, v_0, w_0, and σ_0 from Eqs. (4-9) yields two independent compatibility equations, ϕ_x and ϕ_y, in the x and y directions, respectively:

$$\phi_x = u_1 - u_2 + h_1 w_{1,x} + h_2 w_{2,x} + c^3 (G_x \gamma_{x,xx} + G_y \gamma_{y,xy})/12E - (c + 2G_x/K_x)\,\gamma_x \tag{12}$$

$$\phi_y = v_1 - v_2 + h_1 w_{1,y} + h_2 w_{2,y} + c^3 (G_x \gamma_{x,xy} + G_y \gamma_{y,yy})/12E - (c + 2G_y/K_y)\,\gamma_y \tag{13}$$

The total system potential energy Π consists of the contributions from both faces U_1 and U_2, core shear strain U_3, core normal strain U_4, and the potential term V due to applied forces. The effect of the nonrigid adhesive layer is considered by introducing Lagrange multipliers λ_x and λ_y, which denote the interface shear stresses in the x and y directions, respectively, into the total potential energy contributions.

Thus,

$$\Pi = U_1 + U_2 + U_3 + U_4 + \int_0^a\int_0^b \lambda_x \phi_x \,dy\,dx + \int_0^a\int_0^b \lambda_y \phi_y \,dy\,dx + V \tag{14a}$$

The detailed expressions of each contribution terms are

$$\begin{aligned} U_i = \frac{1}{2}\int_0^a\int_0^b \{ & A_{11}^i (u_{i,x})^2 + 2A_{12}^i u_{i,x} v_{i,y} + A_{22}^i (v_{i,y})^2 \\ & + 2A_{16}^i (u_{i,x} u_{i,y} + u_{i,x} v_{i,x}) \\ & + 2A_{16}^i (v_{i,y} u_{i,y} + v_{i,y} v_{i,x}) + A_{66}^i (u_{i,y} + v_{i,x})^2 \\ & - 2B_{11}^i u_{i,x} w_{i,xx} - 2B_{22}^i v_{i,y} w_{i,yy} \\ & - 2B_{12}^i (v_{i,y} w_{i,xx} + u_{i,x} w_{i,yy}) \\ & - 2B_{16}^i (u_{i,y} w_{i,xx} + v_{i,x} w_{i,xx} + 2u_{i,x} w_{i,xy}) \\ & - 2B_{26}^i (u_{i,y} w_{i,yy} + v_{i,x} w_{i,yy} + 2v_{i,y} w_{i,xy}) \\ & - 4B_{66}^i (u_{i,y} w_{i,xy} + v_{i,x} w_{i,xy}) \\ & + D_{11}^i (w_{i,xx})^2 + 2D_{12}^i w_{i,xx} w_{i,yy} + D_{22}^i (w_{i,yy})^2 \\ & + 4D_{16}^i w_{i,xx} w_{i,xy} + 4D_{26}^i w_{i,yy} w_{i,xy} \\ & + 4D_{66}^i (w_{i,xy})^2 \}\,dy\,dx \end{aligned} \tag{14b}$$

where $i = 1$ for upper face and $i = 2$ for lower face. The core shear contribution is

$$U_3 = (c/2)\int_0^a\int_0^b (G_x \gamma_x^2 + G_y \gamma_y^2)\,dy\,dx \tag{14c}$$

The core transverse normal strain contribution is

$$U_4 = \int_0^a\int_0^b [c^3 (G_x \gamma_{x,x} + G_y \gamma_{y,y})^2/24E + E(w_1 - w_2)^2/2c]\,dy\,dx \tag{14d}$$

The work done by the inplane normal and shear loading is

$$\begin{aligned} V = \frac{1}{4}\int_0^a\int_0^b \{ & N_x (w_{1,x})^2 + N_y (w_{1,y})^2 + 2N_{xy} w_{1,x} w_{1,y} \\ & + N_x (w_{2,x})^2 + N_y (w_{2,y})^2 + 2N_{xy} w_{2,x} w_{2,y} \}\,dy\,dx \end{aligned} \tag{14e}$$

The Rayleigh-Ritz procedure requires the proper selection of the trial functions that satisfy at least the essential boundary conditions and inherently as many imposed natural boundary conditions as possible. For simply supported sandwich plates, a solution is found in the following forms that satisfy all the essential boundary conditions:

$$\begin{aligned} u_1 &= \sum_m \sum_n C_{mn}^1 \cdot \cos\alpha_m x \cdot \sin\beta_n y \\ u_2 &= \sum_m \sum_n C_{mn}^2 \cdot \cos\alpha_m x \cdot \sin\beta_n y \\ v_1 &= \sum_m \sum_n C_{mn}^3 \cdot \sin\alpha_m x \cdot \cos\beta_n y \\ v_2 &= \sum_m \sum_n C_{mn}^4 \cdot \sin\alpha_m x \cdot \cos\beta_n y \\ w_1 &= \sum_m \sum_n C_{mn}^5 \cdot \sin\alpha_m x \cdot \cos\beta_n y \\ w_2 &= \sum_m \sum_n C_{mn}^6 \cdot \sin\alpha_m x \cdot \cos\beta_n y \\ \gamma_x &= \sum_m \sum_n C_{mn}^7 \cdot \cos\alpha_m x \cdot \sin\beta_n y \\ \gamma_y &= \sum_m \sum_n C_{mn}^8 \cdot \sin\alpha_m x \cdot \cos\beta_n y \\ \lambda_x &= \sum_m \sum_n C_{mn}^9 \cdot \cos\alpha_m x \cdot \sin\beta_n y \\ \lambda_y &= \sum_m \sum_n C_{mn}^{10} \cdot \sin\alpha_m x \cdot \cos\beta_n y \end{aligned} \tag{15}$$

with

$$\alpha_m = m\pi/a$$

$$\beta_n = n\pi/b$$

where m and n are half-wavelength integers and C_{mn}^1–C_{mn}^{10} are undetermined coefficients.

The natural boundary conditions are not fully satisfied because of the presence of the coupling terms such as A_{16}^i, A_{26}^i, B_{16}^i, B_{26}^i, D_{16}^i, and D_{26}^i. By retaining a sufficiently large number of terms that are capable of representing the deformed shapes of the unknown variables in Eq. (15), the shortcoming of not satisfying the natural boundary conditions in full can be compensated.

Putting the given assumed forms into the total energy expression Π, it is to be minimized with respect to the undetermined coefficients. Taking variation of the potential energy expression with respect to the basic undetermined coefficients C_{mn}^1–C_{mn}^{10} yields to 10 recurring simultaneous equations.

For $\dfrac{\partial \Pi}{\partial C_{mn}^1} = 0$,

$$\begin{aligned} & (A_{11}^1 \alpha_m^2 + A_{66}^1 \beta_n^2)\, C_{mn}^1 + (A_{12}^1 + A_{66}^1)\, \alpha_m \beta_n C_{mn}^3 \\ & - [(B_{12}^1 + 2B_{66}^1)\, \alpha_m \beta_n^2 + B_{11}^1 \alpha_m^3]\, C_{mn}^5 + C_{mn}^9 \\ & + \sum_p \sum_q \{ - A_{16}^1 (\alpha_m \beta_q L^{mn} + \beta_n \alpha_p L^{pq})\, C_{pq}^1 \\ & - (A_{16}^1 \alpha_m \alpha_p L^{mn} + A_{26}^1 \beta_n \beta_q L^{pq})\, C_{pq}^3 + [(B_{16}^1 \alpha_p^2 \\ & + B_{26}^1 \beta_q^2)\, \beta_n L^{pq} + 2B_{16}^1 \alpha_m \alpha_p \beta_q L^{mn}]\, C_{pq}^5 \} = 0 \end{aligned} \tag{16a}$$

For $\dfrac{\partial \Pi}{\partial C_{mn}^2} = 0$,

$$(A^2_{11}\alpha_m^2 + A^2_{66}\beta_n^2)\, C^2_{mn} + (A^2_{12} + A^2_{66})\, \alpha_m\beta_n C^4_{mn}$$
$$- [(B^2_{12} + 2B^2_{66})\, \alpha_m\beta_n^2 + B^1_{11}\alpha_m^3]\, C^6_{mn} - C^9_{mn}$$
$$+ \sum_p\sum_q \{ - A^2_{16}\,(\alpha_m\beta_q L^{mn} + \beta_n\alpha_p L^{pq})\, C^2_{pq}$$
$$- (A^2_{16}\alpha_m\alpha_p L^{mn} + A^2_{26}\beta_n\beta_q L^{pq})\, C^4_{pq} + [(B^2_{16}\alpha_p^2$$
$$+ B^2_{26}\beta_q^2)\, \beta_n L^{pq} + 2B^2_{16}\alpha_m\alpha_p\beta_q L^{mn}]\, C^6_{pq} \} = 0 \qquad (16b)$$

For $\dfrac{\partial \Pi}{\partial C^3_{mn}} = 0$,

$$(A^1_{12} + A^1_{66})\, \alpha_m\beta_n C^1_{mn} + (A^1_{66}\alpha_m^2 + A^1_{22}\beta_n^2)\, C^3_{mn}$$
$$- [(B^1_{12} + 2B^1_{66})\, \alpha_m^2\beta_n + B^1_{22}\beta_n^3]\, C^5_{mn} + C^{10}_{mn}$$
$$+ \sum_p\sum_q \{ - (A^1_{16}\alpha_m\alpha_p L^{pq} + A^1_{26}\beta_n\beta_q L^{mn})\, C^1_{pq}$$
$$- A^1_{26}\,(\beta_n\alpha_p L^{mn} + \alpha_m\beta_q L^{pq})\, C^3_{pq} + [(B^1_{16}\alpha_p^2$$
$$+ B^1_{26}\beta_q^2)\, \alpha_m L^{pq} + 2B^1_{26}\beta_n\alpha_p\beta_q L^{mn}]\, C^5_{pq} \} = 0 \qquad (16c)$$

For $\dfrac{\partial \Pi}{\partial C^4_{mn}} = 0$,

$$(A^2_{12} + A^2_{66})\, \alpha_m\beta_n C^2_{mn} + (A^2_{66}\alpha_m^2 + A^2_{22}\beta_n^2)\, C^4_{mn}$$
$$- [(B^2_{12} + 2B^2_{66})\, \alpha_m^2\beta_n + B^2_{22}\beta_n^3]\, C^6_{mn} - C^{10}_{mn}$$
$$+ \sum_p\sum_q \{ - (A^2_{16}\alpha_m\alpha_p L^{pq} + A^2_{26}\beta_n\beta_q L^{mn})\, C^2_{pq}$$
$$- A^2_{26}\,(\beta_n\alpha_p L^{mn} + \alpha_m\beta_q L^{pq})\, C^4_{pq} + [(B^2_{16}\alpha_p^2$$
$$+ B^2_{26}\beta_q^2)\, \alpha_m L^{pq} + 2B^2_{26}\beta_n\alpha_p\beta_q L^{mn}]\, C^6_{pq} \} = 0 \qquad (16d)$$

For $\dfrac{\partial \Pi}{\partial C^5_{mn}} = 0$,

$$- [B^1_{11}\alpha_m^3 + (B^1_{12} + 2B^1_{66})\, \alpha_m\beta_n^2]\, C^1_{mn}$$
$$- [B^1_{22}\beta_n^3 + (B^1_{12} + 2B^1_{66})\, \alpha_m^2\beta_n] - C^3_{mn}$$
$$+ [D^1_{11}\alpha_m^4 + 2(D^1_{12} + 2D^1_{66})\, \alpha_m^2\beta_n^2 + D^1_{22}\beta_n^4 + E/c]\, C^5_{mn}$$
$$- (E/c)\, C^6_{mn} + h_1\alpha_m C^9_{mn} + h_1\beta_n C^{10}_{mn}$$
$$+ \sum_p\sum_q [(B^1_{16}\alpha_m^2 + B^1_{26}\beta_n^2)\, \beta_q L^{mn} + 2B^1_{16}\alpha_m\beta_n\alpha_p L^{pq}]\, C^1_{pq}$$
$$+ \sum_p\sum_q [(B^1_{16}\alpha_m^2 + B^1_{26}\beta_n^2)\, \alpha_p L^{mn} + 2B^1_{16}\alpha_m\beta_n\beta_q L^{pq}]\, C^3_{pq}$$
$$- \sum_p\sum_q [2D^1_{16}\,(\alpha_m^2\alpha_p\beta_q L^{mn} + \alpha_m\beta_n\alpha_p^2 L^{pq})$$
$$+ 2D^1_{26}\,(\beta_n^2\alpha_p\beta_q L^{mn} + \alpha_m\beta_n\beta_q^2 L^{pq})]\, C^5_{pq}$$
$$= P_{mn} + ½\,(N_x\alpha_m^2 + N_y\beta_n^2)\, C^5_{mn}$$
$$+ \sum_p\sum_q (½)\, N_{xy}\,(\alpha_m\beta_q L^{pn}_{mq} + \beta_n\alpha_p L^{mq}_{pn})\, C^5_{pq} \qquad (16e)$$

For $\dfrac{\partial \Pi}{\partial C^6_{mn}} = 0$,

$$- [B^2_{11}\alpha_m^3 + (B^2_{12} + 2B^2_{66})\, \alpha_m\beta_n^2]\, C^2_{mn}$$
$$- [B^2_{22}\beta_n^3 + (B^2_{12} + 2B^2_{66})\, \alpha_m^2\beta_n]\, C^4_{mn}$$
$$+ [D^2_{11}\alpha_m^4 + 2(D^2_{12} + 2D^2_{66})\, \alpha_m^2\beta_n^2 + D^2_{22}\beta_n^4 + E/c]\, C^6_{mn}$$
$$- (E/c)\, C^5_{mn} + h_2\alpha_m C^9_{mn} + h_2\beta_n C^{10}_{mn}$$
$$+ \sum_p\sum_q [(B^2_{16}\alpha_m^2 + B^2_{26}\beta_n^2)\, \beta_q L^{mn} + 2B^2_{16}\alpha_m\beta_n\alpha_p L^{pq}]\, C^2_{pq}$$
$$+ \sum_p\sum_q [(B^2_{16}\alpha_m^2 + B^2_{26}\beta_n^2)\, \alpha_p L^{mn} + 2B^2_{26}\alpha_m\beta_n\beta_q L^{pq}]\, C^4_{pq}$$
$$- \sum_p\sum_q [2D^2_{16}\,(\alpha_m^2\alpha_p\beta_p L^{mn} + \alpha_m\beta_n\alpha_p^2 L^{pq})$$
$$+ 2D^2_{26}\,(\beta_n^2\alpha_p\beta_q L^{mn} + \alpha_m\beta_n\beta_q^2 L^{pq})]\, C^6_{pq}$$
$$= ½\,(N_x\alpha_m^2 + N_y\beta_n^2)\, C^6_{mn}$$
$$+ \sum_p\sum_q (½)\, N_{xy}\,(\alpha_m\beta_q L^{pn} + \beta_n\alpha_p L^{mq})\, C^6_{pq} \qquad (16f)$$

For $\dfrac{\partial \Pi}{\partial C^7_{mn}} = 0$,

$$\left(1 + \frac{c^2G_x\alpha_m^2}{12E}\right) C^7_{mn} + \left(\frac{c^2G_y\alpha_m\beta_n}{12E}\right) C^8_{mn}$$
$$- \left(\frac{c^2\alpha_m^2}{12E} + \frac{1}{G_x} + \frac{2}{cK_x}\right) C^9_{mn} - \left(\frac{c^2\alpha_m\beta_n}{12E}\right) C^{10}_{mn} = 0 \qquad (16g)$$

For $\dfrac{\partial \Pi}{\partial C^8_{mn}} = 0$,

$$\left(\frac{c^2G_x\alpha_m\beta_n}{12E}\right) C^7_{mn} + \left(1 + \frac{c^2G_y\beta_n^2}{12E}\right) C^8_{mn}$$
$$- \left(\frac{c^2\alpha_m\beta_n}{12E}\right) C^9_{mn} - \left(\frac{c^2\beta_n^2}{12E} + \frac{1}{G_y} + \frac{2}{cK_y}\right) C^{10}_{mn} = 0 \qquad (16h)$$

For $\dfrac{\partial \Pi}{\partial C^9_{mn}} = 0$,

$$C^1_{mn} - C^2_{mn} + h_1\alpha_m C^5_{mn} + h_2\alpha_m C^6_{mn} - \left(\frac{c^3G_x\alpha_m^2}{12E} + c\right.$$
$$\left. + \frac{2}{cK_y}\right) C^7_{mn} - \left(\frac{c^3G_y\alpha_m\beta_n}{12E}\right) C^8_{mn} = 0 \qquad (16i)$$

For $\dfrac{\partial \Pi}{\partial C^{10}_{mn}} = 0$,

$$C^3_{mn} - C^4_{mn} + h_1\beta_n C^5_{mn} + h_2\beta_n C^6_{mn} - \left(\frac{c^3G_x\alpha_m\beta_n}{12E}\right) C^7_{mn}$$
$$- \left(\frac{c^3G_y\beta_n^2}{12E} + c + \frac{2G_y}{K_y}\right) C^8_{mn} = 0 \qquad (16j)$$

with

$$L^{mn} = \frac{16\, mn}{\pi^2(m^2 - p^2)\,(n^2 - q^2)}$$
$$L^{pq} = \frac{16\, pq}{\pi^2(m^2 - p^2)\,(n^2 - q^2)}$$
$$L^{pn} = \frac{-16\, pn}{\pi^2(m^2 - p^2)\,(n^2 - q^2)}$$
$$L^{mq} = \frac{-16\, mq}{\pi^2(m^2 - p^2)\,(n^2 - q^2)} \qquad (17)$$

where m, n, p, and q are chosen such that $m + p$ and $n + q$ are odd integers. The stability criterion of a sandwich plate reduces to the generalized eigenvalue problem of type $[A]\{x\} = N_x[B]\{x\}$.

In the computational procedure, C^1_{mn}–C^4_{mn} and C^7_{mn}–C^{10}_{mn} are eliminated from the whole set of algebraic equations to result in a reduced set of algebraic equations in terms of C^5_{mn} and C^6_{mn}. Usually the reduced set is more appropriate for handling the eigenvalue problem. In the present investigation, the IMSL subroutine EIGZF was used with a Prime minicomputer. The first 16 terms in the series of Eq. (15) are retained, i.e., m, $n = 1,2,3,4$ for the calculation.

Results and Discussion

The buckling loads of anisotropic sandwich plates are compared with Rao[11] for various lay-up angles θ. The data used for comparison and taken from this reference are 1) $a/b = 1$, $a = 225$ mm, and single layered faces, $N = 1$, and 2) carbon/epoxy-layered faces and aluminum core with material properties as listed in Table 1. The present formulation can handle the unbalanced anisotropic sandwich plate with unsymmetric faces. While Rao can solve the balanced sandwich plate with thin and symmetric faces, he did not take the local bending stiffness of faces into account. The present solution is compared with Rao in Table 2.

For the numerical evaluation of the buckling behavior of an anisotropic sandwich plate, one with graphite/epoxy and glass/epoxy hybrid laminated faces and aluminum honeycomb core under inplane normal and shear forces is investigated. The principal material properties are listed in Table 1.

When graphite/epoxy faces are bonded to an aluminum honeycomb core, the bonding strength is degraded due to the galvanic corrosion. Therefore, the glass/epoxy layer can be inserted between the graphite/epoxy face and an aluminum honeycomb core to prevent the degradation in the bonding strength. The unsymmetry of faces is due to the insertion of glass/epoxy layers in the bonding of graphite/epoxy faces to an aluminum honeycomb core. As a typical example that shows all the anisotropic coupling actions, a sandwich plate with layups $[\theta_3{}^{gr/ep}/(\theta\text{–}90)^{gl/ep}/\text{core}/(90\text{–}\theta)^{gl/ep}/(-\theta)_3{}^{gr/ep}]$ is considered.

Table 3 shows the normalized anisotropic coupling terms for the upper face of the sandwich plate under consideration and the buckling loads of the anisotropic sandwich plates and the orthotropic plates in which stretching-shearing coupling, bending-stretching coupling, and bending-twisting coupling are neglected for the various values of θ. Numerical results for the square sandwich plates under uniaxial compression are shown in Figs. 3 and 4. The angle θ between the plate edges (x,y) and the face principal elastic axes $(1,2)$ serves as a convenient parameter for defining the anisotropy of face material. Figure 3 shows the effect of angle θ upon the buckling load ratio of anisotropic plate to orthotropic plate, N_x/N_x^o, for three ratios of core to face thickness c/d. For thick-faced sandwich plates ($c/d = 5$), the bending-stretching coupling action shows overwhelming effect over the other coupling actions on a buckling behavior. At $\theta = 0$ and 90 deg, B_{11}^i, B_{12}^i, B_{22}^i, and B_{66}^i are the nonzero coupling terms. These coupling terms increase the buckling resistance at $\theta = 0$ deg and increases at $\theta = 90$ deg in comparison with orthotropic plates, as observed by Monforton and Ibrahim.[9] The coupling effect in the thick-faced sandwich plate ($c/d < 10$) is quite different

Table 1 Material properties of faces and core for elastic analysis of anisotropic sandwich plates

Face	E_1, GPa	E_2, GPa	ν_{12}	G_{12}, GPa	t_{ply}, mm
Carbon/epoxy[11]	229.00	13.35	0.315	5.25	0.200
Graphite/epoxy	181.00	10.30	0.280	7.17	0.125
Glass/epoxy	38.6	8.27	0.260	4.14	0.125
Core	**E, GPa**	**G_x, GPa**	**G_y, GPa**	**c, mm**	
Soft ($C_r = 0.1$)	0.02	0.0146	0.00904	10.0	
Typical[11] ($C_r = 1.0$)	0.20	0.1460	0.09040	10.0	
Stiff ($C_r = 10$)	2.00	1.4600	0.90400	10.0	

Table 2 Buckling of sandwich plates with single-layered faces under uniaxial compressive load, $[\theta/\text{core}/\theta]$, $a = 225$ mm, $a/b = 1$, $c/d = 50$, $c = 10$ mm, $N = 1$

Angle, deg	Rao, N/mm	Present[a] N/mm
0	423.8	427.6
10	428.7	432.4
20	443.5	447.0
30	459.5	463.4
40	468.7	472.4
45	–	467.0
50	–	437.9
60	356.6	360.1
70	285.6	288.4
80	233.6	235.6
90	213.8	215.6

[a] $K = 10^5$ MPA/mm, $E = 0.2$ MPa.

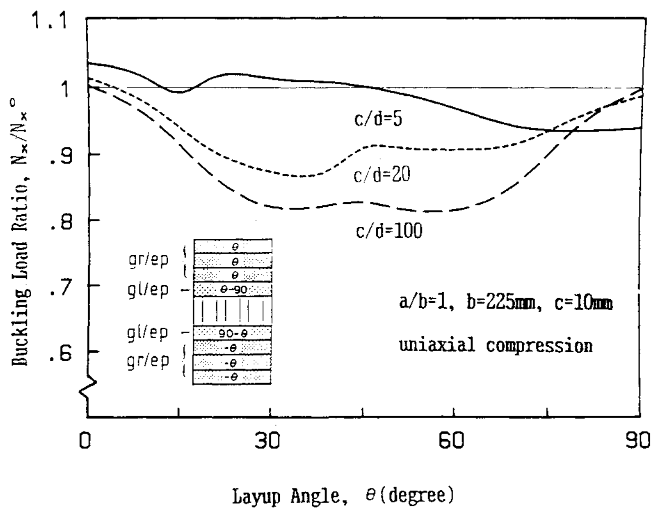

Fig. 3 Effect of lay-up angle θ on the buckling load ratio of anisotropic plate to orthotropic plate, N_x/N_x^o.

Table 3 Buckling load of sandwich plates with general laminated faces and stiffness components of the upper face under uniaxial compressive load, $[0_3^{gr/ep}/(\theta\text{–}90)^{gl/ep}/\text{core}/(90\text{–}\theta)^{gl/ep}/(-\theta)_3^{gr/ep}]$, $a = 225$ mm, $a/b = 1$, $c/d = 20$, $c = 10$ mm

θ deg	0	10	20	30	40	45	50	60	70	80	90
N_x, N/mm	705.3	724.4	774.1	834.1	880.4	851.3	785.9	657.9	560.1	499.0	476.6
N_x^o, N/mm	695.8	744.2	853.6	955.6	1002.	935.8	863.1	725.4	611.4	521.3	482.6
A_{11}^*, MPa	1.385	1.321	1.114	0.848	0.582	0.468	0.372	0.244	0.188	0.175	0.176
A_{22}^*, MPa	0.176	0.175	0.188	0.244	0.372	0.468	0.582	0.848	1.114	1.321	1.385
A_{16}^{**}	0.000	0.424	0.761	0.874	0.754	0.646	0.525	0.278	0.089	0.006	0.000
B_{11}^{**}	0.660	0.639	0.558	0.406	0.234	0.159	0.093	– 0.012	– 0.077	– 0.104	– 0.110
B_{16}^{**}	0.000	0.109	0.199	0.239	0.223	0.203	0.178	0.122	0.067	0.026	0.000
D_{16}^{**}	0.000	0.316	0.561	0.634	0.531	0.444	0.347	0.156	0.022	– 0.020	0.000

where $[A^*] = [A]/d$, $[B^*] = 2[B]/d^2$, $[D^*] = 12[D]/d^3$

and $A_{16}^{**} = A_{16}^*/\sqrt{A_{11}^* \cdot A_{22}^*}$, $B_{16}^{**} = B_{16}^*/\sqrt{A_{11}^* \cdot A_{22}^*}$, $D_{16}^{**} = D_{16}^*/\sqrt{A_{11}^* \cdot A_{22}^*}$

from that in the thin-faced sandwich plate ($c/d > 10$), which is mainly due to the relative strength of coupling terms. Figure 4 shows the effects of c/d on the buckling load ratio for various face layups. For unsymmetric crossply faced sandwich plates ($\theta = 0$ and 90 deg), the bending-stretching coupling effect decreases continuously as the c/d value increases. For the sandwich plates with the other face layups, the bending-stretching coupling effect diminishes, but the stretching-shearing coupling effect gets stronger as the c/d value increases.

The effect of aspect ratio a/b on the buckling load of the anisotropic sandwich plate of $c/d = 20$ in comparison with that of the orthotropic sandwich plate is shown in Fig. 5 for the range of a/b from 0.5 to 2.0. For $\theta = 0$ deg, the buckling load of an anisotropic plate is a little greater than that of an orthotropic one when $a/b < 1.5$, but the reverse is true when $a/b > 1.5$. For the layups other than $\theta = 0$ deg, the buckling load of an anisotropic plate is much smaller than that of an orthotropic one due to the presence of the coupling actions, especially the stretching-shearing coupling. The effect of lay-up angle θ on the buckling loads of sandwich plates with $c/d = 20$ is shown in Figs. 6 and 7 for various aspect ratios and inplane loading conditions. The dependence of the buckling resistance on the lay-up angle θ is quite different for the given loading conditions and various aspect ratios. For $a/b = 1$, the shear buckling load is relatively insensitive to the variation of θ. For $a/b = 0.5$ and 2.0, the shear buckling load is maximum when the reinforcing fiber is parallel to the shorter side of the plate (Fig. 6). For a square sandwich plate under compressive loading, the buckling resistance is maximum when θ is about 40 deg. The compressive buckling load is sensitive to the variation of aspect ratio for the lay-up angle around $\theta = 0$ deg, but the opposite is true for the lay-up angle around $\theta = 0$ deg. The lay-up angle that yields the maximum buckling resistance to the biaxial compression moves to the larger value as a/b increases (Fig. 7).

The effect of core properties on the buckling load of a sandwich plate for various c/d values and bonding stiffnesses is shown in Fig. 8. The abscissa of the figure is taken as a multiple of the listed typical core properties, which is defined as the core properties ratio C_r. In sandwich structures, the mechanical properties of a core seriously affect the magnitude of the buckling resistance, but these are also dependent upon the density of a core. For the convenience of the parametric study, it is assumed that the bonding stiffnesses in x and y directions are the same, i.e., $K_x = K_y = K$. For a sandwich with thin faces, the change in the bonding stiffness K has negligible effects on the buckling resistance. The buckling resistance is affected by the change of K for a sandwich with thick faces.

The effect of the adhesive bonding stiffness K on the buckling load of the typical sandwich plate is shown in Fig. 9 for three ratios of core properties, C_r and $c/d = 20$. As K

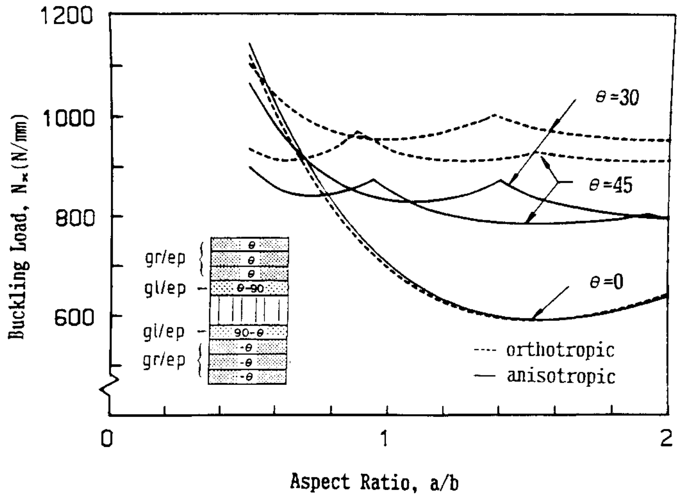

Fig. 5 Effect of aspect ratio a/b on the buckling load under uniaxial compression.

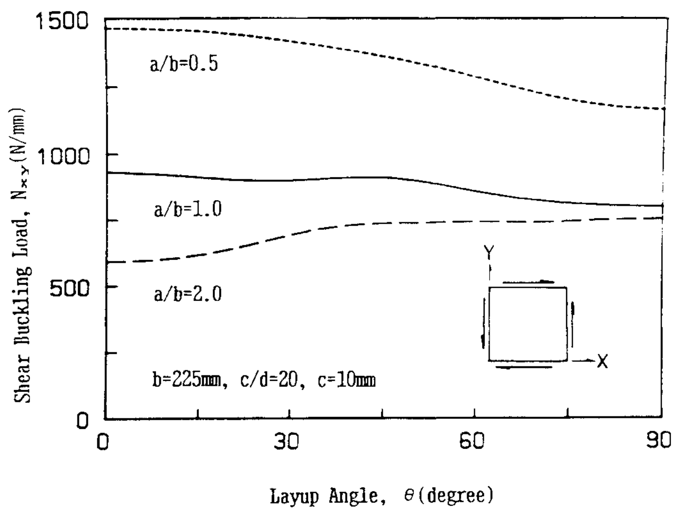

Fig. 6 Effect of lay-up angle θ on the buckling loads under shear loading.

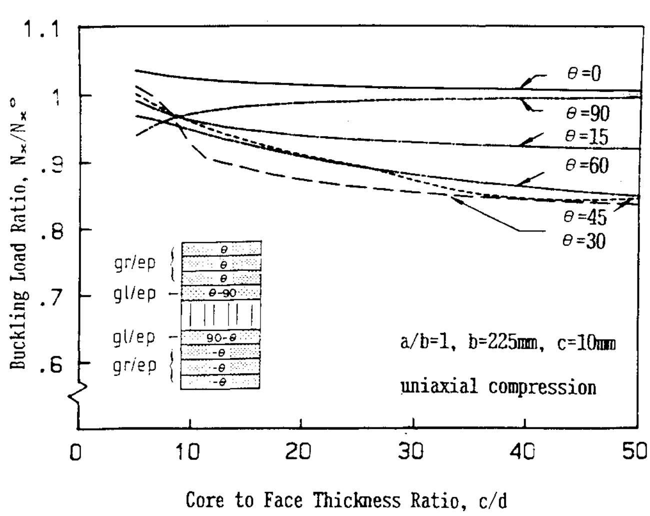

Fig. 4 Effect of the thickness ratio c/d on the buckling load ratio of anisotropic plate to orthotropic plate, N_x/N_x^o.

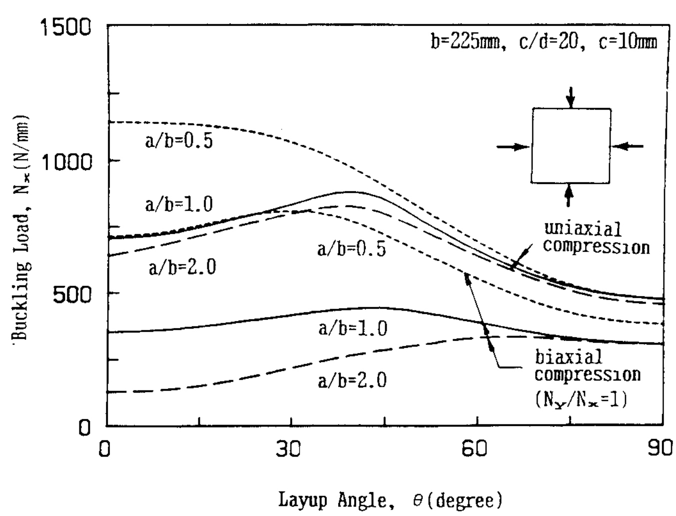

Fig. 7 Effect of lay-up angle θ on the buckling loads under compressive loading.

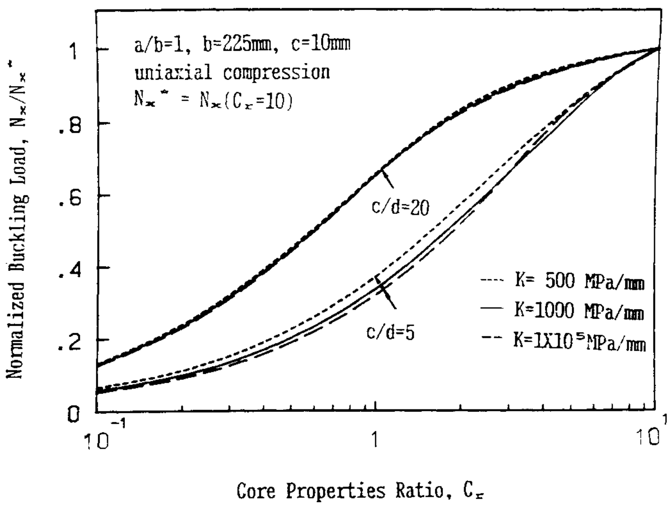

Fig. 8 Effect of core properties ratio C_r on the buckling load under uniaxial compression (N_x^* = buckling load for C_r = 10).

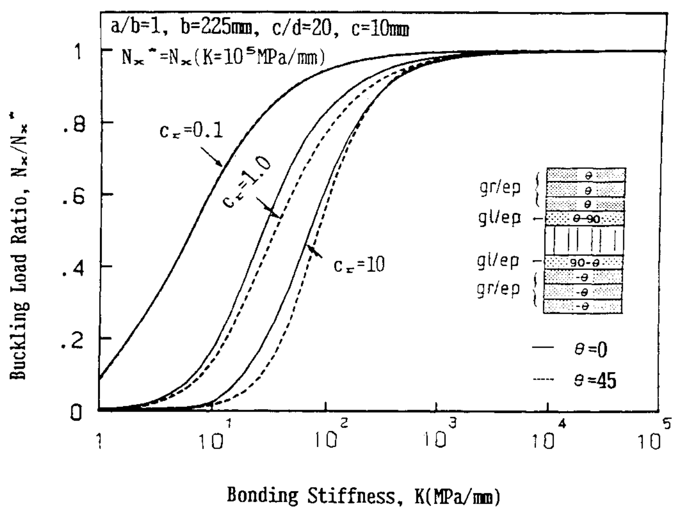

Fig. 9 Effect of bonding stiffness K on the buckling load under uniaxial compression (N_x^* = buckling load for $K = 10^5$ MPa/mm).

becomes larger than 5×10^3 MPa/mm, the buckling load approaches that of a perfectly bonded sandwich plate. When the bonding stiffness is smaller than the threshold value, a slight decrease in the bonding stiffness will cause a serious reduction in the buckling resistance, especially for a sandwich plate with a stiff core. It can be deduced that the bonding stiffness is quite critical for a sandwich plate with thick faces and stiff core when it is smaller than the threshold value.

Conclusion

The buckling of unbalanced simply supported anisotropic sandwich plates with finite bonding stiffness is analyzed, considering bending-stretching, stretching-shearing, and bending-twisting coupling actions simultaneously by the Rayleigh-Ritz method. The numerical results show that the buckling load ratio of anisotropic to orthotropic plate is dependent upon the core to face thickness ratio and lay-up angles. The thick-faced sandwich plates show different coupling effects from the thin-faced ones due to the relative strength of the coupling actions. The perfect bonding assumption is reasonable for a sandwich with higher bonding stiffness than a threshold value, but the bonding stiffness is critical for a sandwich plate with thick faces and stiffness core when it is smaller than the threshold value.

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